On models of non-Eucledian spaces generated by associative algebras

Maria Trnková[1,2]

email: M.D.Trnkova@gmail.com

[1]*Department of Algebra and Geometry, Faculty of Science, Palacky University in Olomouc, Tomkova 10, Olomouc, Czech Republic*

[2]*Department of Geometry, NIIMM, Kazan State University, prof. Nuzhina 1/37, Kazan, Russia*

Abstract *We present the non-trivial example how to generate non-Euclidean geometries from associative unital algebras. We consider bundles of the sphere of the degenerate non-Eucleadian space and its two models. The first (conformal) model is obtained by the mapping S onto a plane pass through the origin. It is analogous to the stereographic mapping. The second model (projective) is constructed by the Norden normalization method, where we project the sphere onto a plane of normalization defining the metric and Christoffel symbols which allow us to find geodesic curves.*

1. Introduction

A lot of models of non-Eucledian spaces were studied in the past, especially spaces of a constant curvature, projective spaces and the conformal planes (e.g. [1], [2], [3], [4]). There exists a lot of studies on how these models can be generated by algebras. It is well known that algebras define some structures in bundle manifolds of different types (e.g. [5], [6], [7]). In the literature, we can find many applications of this approach on the cases of non-Eucledian spaces (e.g. [8], [9], [10], [11], [12]).

We would like to present non-standard models within this framework. In the beginning, we describe how an associative algebra generates a vector space and we also discuss some of its properties. In the next section we define a sphere and the map S in this vector space and we use it to construct a conformal model. In the third section we remind some facts about the Norden normalization method [13] and we use it for the construction of a projective model.

Let us denote by $\mathfrak{A}$ a unital associative n-dimensional algebra with the multiplication xy, and by $G \subset \mathfrak{A}$ the set of invertible elements. Then G is a Lie

group with the same multiplication rule. Let $\mathfrak{B} \subset \mathfrak{A}$ be a unital subalgebra of an algebra $\mathfrak{A}$ and $H \subset G$ be the set of its invertible elements. So, H is a Lie subgroup of group G and G/H is the factor-space of right cosets. A bundle $(G, \pi, M = G/H)$ is a principal bundle with the structure group H, where π is a canonical projection (for example, [14], [15]).

Foundations of the theory of finite-dimensional associative algebras were made by E. Cartan (1898), Wedderburn (1908) and F. E. Molin (1983), who discovered the structure of any algebra over an arbitrary base field [16]. E. Study and E. Cartan in [17] classified all 3 and 4-dimensional unital associative irreducible[1] algebras up to an isomorphism. This classification could be also found in [18]. In this paper we consider only one type of 3-dimensional algebra $\mathfrak{A}$. We leave a more complicated 4-dimensional case for a future work.

Let $\{\mathbf{1}, e_1, e_2\}$ be a basis of our algebra $\mathfrak{A}$ with the identity element $\mathbf{1}$. The multiplication rules are:

$$(e_1)^2 = \mathbf{1},\ (e_2)^2 = 0,\ e_1e_2 = -e_2e_1 = e_2. \tag{1}$$

The matrix representation of an algebra $\mathfrak{A}$ is a space of upper triangular matrices $T_u = \{\begin{pmatrix} x_0 & x_2 \\ 0 & x_1 \end{pmatrix} | x = x_0 + x_1 \cdot e_1 + x_2 \cdot e_2 \in \mathfrak{A}\}$ with the basic elements [16]

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

We consider the trivial conjugation $x = x_0 + x^i e_i \to \overline{x} = x_0 - x^i e_i$ with the property $\overline{xy} = \overline{y}\,\overline{x}$ and the bilinear form

$$(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}). \tag{3}$$

This form takes the real values and it determines a degenerate scalar product:

$$(x, y) = x_0y_0 - x_1y_1. \tag{4}$$

It defines a structure of degenerated pseudo-Euclidean vector spaces with rank 2 in the algebra $\mathfrak{A}$. (It is also possible to call this space as "semi-pseudo-Euclidean", but later we will call it just "pseudo-Euclidean".) The set of invertible elements $G = \{x \in \mathfrak{A} \mid (x_0)^2 - (x_1)^2 \neq 0\}$ is a non-Abelian Lie group. Its underlying manifold is $\mathbb{R}^3$ without two transversal 2-planes, hence it consists of 4 connected components.

The norm is defined as usual, $|x, y|^2 = (x - y, x - y)$. The geodesic curves $x(t)$ are then

$$x_0 = a_0t + b_0 \qquad x_1 = a_1t + b_1 \qquad x_2 = f(t)$$

where $f(t)$ is an arbitrary function of t and $a_0,\ a_1,\ b_0,\ b_1$ are the numerical coefficients.

[1] *Irreducible* means indecomposable into a direct sum of algebras.

In the basis (2) we can find two subalgebras: $R(e_1)$ with basis $\{\mathbf{1}, e_1\}$, it is an algebra of double numbers, and a subalgebra $R(e_2)$ with basis $\{\mathbf{1}, e_2\}$, it is an algebra of dual numbers. The set of their invertible elements $H_1 = \{x_0 + x_1 e_1 \in R(e_1) \mid x_0^2 - x_1^2 \neq 0\}$ and $H_2 = \{x_0 + x_2 e_2 \in R(e_2) \mid x_0 \neq 0\}$ are Lie subgroups of the Lie group G.

The space of right cosets $H_1 x$ defines a trivial principal bundle $(G, \pi, M = G/H_1)$ over the real line $\mathbb{R}$. The fiber is a plane without two transversal lines and the structure group is H_1. The manifold of the group G is diffeomorphic to direct sum $\mathbb{R} \times H_1$. The coordinate view of the canonical projection π is:

$$\pi(x) = \frac{x_2}{x_0 - x_1}. \tag{5}$$

The equation of fibers is:

$$u(x_0 - x_1) - x_2 = 0, \ u \in \mathbb{R}. \tag{6}$$

Let us investigate $\mathbb{G}$, the group of transformations of Lie group G. We can easily find that it has no dilations and inversions while there is a vertical translation $x \to x + a$, $a \in G$. Furthermore, $\mathbb{G}$ includes the rotations, resp. anti-rotations,

$$x' = ax \qquad \text{or} \qquad x' = xa$$

with $|a|^2 = 1$, resp. $|a|^2 = -1$. These elements can be represented as:

$$a = \cosh\varphi \pm \sinh\varphi\, e_1 + u \sinh\varphi\, e_2, \quad \text{resp.} \quad a = \sinh\varphi \pm \cosh\varphi e_1 + u\cosh\varphi\, e_2,$$

where $u \in \mathbb{R}$. The anti-rotations transform the elements with the positive norms into the elements with the negative norms and visa versa.

The bilinear form (3) in the algebra $\mathfrak{A}$ takes the real values, therefore it is possible to present it as: $(x, y) = \frac{1}{2}(x\overline{y} + y\overline{x}) = \frac{1}{2}(\overline{x}y + \overline{y}x)$. Consequently, in the case of rotations the hyperbolic cosine of an angle between x and x' are equal to

$$\cosh(x, x') = \frac{(x, ax)}{|x||ax|} = \frac{1/2(x\overline{a}\overline{x} + ax\overline{x})}{|x|^2} = \frac{1/2(x\overline{x}\,\overline{a} + ax\overline{x})}{|x|^2} = \frac{1}{2}(\overline{a} + a) = \cosh\varphi, \tag{7}$$

and the same for the right multiplication. Similarly we get $\sinh\varphi$ for anti-rotations. Note that the angle φ does not depend on x.

Transformations

$$x' = axb, \tag{8}$$

where $|a|^2 = \pm 1$, $|b|^2 = \pm 1$, are compositions of rotations and/or anti-rotations $x' = ax$ and $x' = xb$. We see that (8) defines *proper* rotations and anti-rotations.

Similarly,

$$x' = a\overline{x}b \tag{9}$$

are compositions of the reflection $x' = \overline{x}$ and transformations (8). These are *improper* rotations and anti-rotations.

Lemma *Any proper or improper rotation/anti-rotation of the pseudo-Euclidean space G can be represented by (8) or (9).*

Proof Rotations and anti-rotations (8), (9) are compositions of odd and even numbers of reflections of planes passing through the origin. To each plane corresponds its orthonormal vector n. If vectors x_1 and n are collinear, then $\overline{x}_1 n = \overline{n} x_1$ and $x'_1 = -n\overline{x}_1 n = -n\overline{n}x_1 = -x_1$. If vectors x_2 and n are orthogonal, then $\overline{x}_2 n + \overline{n} x_2 = 0$ and $x'_2 = -n\overline{x}_2 n = n\overline{n} x_2 = x_2$. On the other hand, any vector x can be represented as a sum of vectors x_1 and x_2. It means, that a reflection of the plane is: $x' = -n\overline{x}n$. Therefore, the composition of even, resp. odd number of reflections of planes are transformation (8), resp. (9). □

Translations and rotations/anti-rotations are then isometries. All transformations can be written in a known form (for further discussion see e.g. [4])

$$\begin{cases} x'_0 = x_0 \cosh\varphi + x_1 \sinh\varphi + a_0 \\ x'_1 = x_1 \cosh\varphi + x_0 \sinh\varphi + a_1 \\ x'_2 = u_0 x_0 + u_1 x_1 + u_2 x_2 + a_2 \end{cases} \tag{10}$$

where $a = a_i e_i \in G$ and $u_i \in \mathbb{R}$.

Let us introduce adapted coordinates (u, λ, φ) of the bundle in semi-Euclidean space, here u is a basic coordinate, λ, φ are fiber coordinates. If $|x|^2 > 0$, we denote $\lambda = \pm\sqrt{x_0^2 - x_1^2} \neq 0$, the sign of λ is equal to the sign of x_0. The adapted coordinates of the bundle in this case are:

$$x_0 = \lambda \cosh\varphi, \quad x_1 = \lambda \sinh\varphi, \quad x_2 = u\lambda \exp\varphi, \tag{11}$$

where $\lambda \in \mathbb{R}_0, \quad u, \varphi \in \mathbb{R}$.

If $|x|^2 < 0$, then we write $\lambda = \pm\sqrt{x_1^2 - x_0^2}$, the sign of λ is equal to the sign of x_1:

$$x_0 = \lambda \sinh\varphi, \quad x_1 = \lambda \cosh\varphi, \quad x_2 = u\lambda \exp\varphi. \tag{12}$$

The structure group acts as follows:

$$u' = u, \quad \lambda' = \lambda\rho, \quad \varphi' = \varphi + \psi, \tag{13}$$

where the element $a(0, \rho, \psi)$ of the structure group acts on the element $x(u, \lambda, \varphi) \in G$. This group consists of 4 connected components.

2. Conformal model of a sphere

We call *semi-Euclidean sphere with an unit radius* the set of all elements of algebra $\mathfrak{A}$ whose square is equal to one,

$$S^2(1) = \{x \in \mathfrak{A} \mid x_0^2 - x_1^2 = 1\}.$$

Analogously, the set of elements with an imaginary unit module $|x|^2 = -1$ we call *semi-Euclidean sphere with an imaginary unit radius* $S^2(-1)$. One of these spheres can be obtained from another one by the rotation.

The transformations (10) are now constrained by additional relation $x_0^2 - y_0^2 = 1$, therefore, only rotations and vertical translations remain, $a_0 = a_1 = 0$.

We consider the subbundle of the bundle $(G, \pi, M = G/H_1)$ of semi-Euclidean sphere $S^2(1)$, i.e. the bundle $\pi : S^2(1) \to M$. The fibers of the new bundle

are intersections of $S^2(1)$ and planes (6). The restriction of the group of double numbers H_1 to $S^2(1)$ is a Lie subgroup S_1 of double numbers with an unit module

$$S_1 = \{a_0 + a_1 e_1 \in H_1 \mid a_0^2 - a_1^2 = 1\}\,.$$

This group consists of two connected components. The bundle $(S^2(1), \pi, M)$ is a principal bundle of the group $S^2(1)$ by the Lie subgroup S_1 to right cosets.

We define coordinates adapted to the bundle on semi-Euclidean sphere $S^2(1)$. If $x \in S^2(1)$ then from (11) we get $\lambda = \varepsilon$, $\varepsilon = \pm 1$. The parametric equation of semi-Euclidean sphere in the adapted coordinates (u, φ) is:

$$\mathbf{r}(u, \varphi) = \varepsilon(\cosh\varphi, \sinh\varphi, u\exp\varphi), \tag{14}$$

where u is a basis coordinate, φ is a fiber coordinate. Different values of ε correspond to different connected components of semi-Euclidean sphere $S^2(1)$.

Let us define the action of the structure group S_1 on semi-Euclidean sphere. From (13) and using the adapted coordinates of elements $a(0, \varepsilon_1, \psi)$, $x(u, \varepsilon, \varphi) \in S^2(1)$ we get:

$$u' = u, \quad \varepsilon' = \varepsilon\varepsilon_1, \qquad \varphi' = \varphi + \psi.$$

This group also consists of two connected components.

The metric tensor for semi-Euclidean sphere has the matrix representation:

$$(g_{ij}) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

The linear element of the metric is:

$$ds_1^2 = -d\varphi^2. \tag{15}$$

Now, we want to define the conformal model of the bundle $(S^2(1), \pi, \mathbb{R})$. For that we need to introduce the conformal map of the sphere to a disconnected plane $f : S^2(1) \to Q \in \mathbb{R}^2$. Q is located at $x_0 = 0$. We know that the sphere consists of two disconnected components, one with $x_0 > 0$, and other with $x_0 < 0$. We choose a pole at the first one, $N(1, 0, 0)$. All points of $S^2(1)$ except the line through the pole N are stereographically projected to Q such that the first component of the sphere with $x_0 > 0$ is mapped on $x_1 = (-\infty, -1) \cup (1, \infty)$ while the second component with $x_0 < 0$ is mapped on the strip $x_1 = (-1, 1)$. We denote x, y coordinates on Q such that the x axis lies along x_1 while the y axis along x_2. Then

$$x = \frac{x_1}{1 - x_0}, \quad y = \frac{x_2}{1 - x_0}\,, \tag{16}$$

An inverse map $f^{-1} : \mathbb{R}^2 \to S^2(1)$ where $x \neq \pm 1$ is:

$$x_0 = -\frac{1 + x^2}{1 - x^2}, \quad x_1 = \frac{2x}{1 - x^2}, \quad x_2 = \frac{2y}{1 - x^2}\,. \tag{17}$$

If we substitute formulas (16) into (14) then we obtain the relations between coordinates x, y and adapted coordinates u, φ which are on semi-Euclidean sphere:

$$f: \quad x = \frac{\sinh\varphi}{\varepsilon - \cosh\varphi}, \quad y = \frac{u\exp\varphi}{\varepsilon - \cosh\varphi}\,.$$

Then the inverse map is:

$$\varphi = \ln\Big(\varepsilon\frac{x-1}{x+1}\Big), \quad u = -\frac{2y}{(1-x)^2}\,. \tag{18}$$

Note that the lines $x = \pm 1$ are not included in the mapping and Q consists of three disconnected components. Also, the line $x_0 = 1, x_1 = 0$ has no image in this mapping. We add it by hand, identifying the image of this line with the points $x = \pm\infty$ on Q. Then two disconnected parts $x = (-\infty, -1)$ and $x = (1, \infty)$ are connected and we call this plane C^2.

In particular, after enlarging Q into C^2 by the infinitely distant point and ideal line crossing this point, then the stereographic map f becomes diffeomorphism S. Note that the infinitely distant point is the image of point N. The ideal line is the image of the straight line belonging to $S^2(1)$ and crossing the pole: $x_0 = 1$, $x_1 = 0$.

Let us now consider the commutative diagram:

$$\begin{array}{ccc} S^2(1) & \xrightarrow{S} & C^2 \\ & \pi \searrow \quad \swarrow p & \\ & \mathbb{R} & \end{array}$$

The map $p = \pi \circ S^{-1} : C^2 \to R$ is defined by this diagram. We find the coordinate form of this map:

$$u = -\frac{2y}{(1-x)^2}\,.$$

The map $p : C^2 \to \mathbb{R}$ defines the trivial principal bundle with the base $\mathbb{R}$ and the structure group S_1.

Theorem *Let S is the map : $S^2(1) \to C^2$ as described before. Then S is a conformal map.*

Proof The metric on G induces the metric on C^2. In the coordinates x, y it has the form:

$$d\tilde{s}^2 = -dx^2. \tag{19}$$

Let us find the metric of semi-Euclidean sphere from the metric on C^2. From (18) we get $d\varphi = \frac{2}{x^2-1}dx$ and using (15) and (19) we find:

$$ds_1^2 = \frac{4}{(x^2-1)^2}d\tilde{s}^2.$$

Hence, the linear element of semi-Euclidean sphere differs from the linear element of C^2 by a conformal factor and therefore, the map S is conformal. $\square$

We find the equation of fibers on C^2. The 1-parametric fibers family of the bundle $(S^2(1), \pi, \mathbb{R})$ in the adaptive coordinates (14) is: $u = c, \quad c \in \mathbb{R}$. From (18) we get the image of this family under the map S:

$$y = -c/2 \cdot (x-1)^2. \tag{20}$$

The C^2 plane is also fibred by this 1-parametric family of parabolas.

3. The projective conformal model

Now we construct the projective semi-conformal model of the sphere $S^2(1)$ and the principal bundle on it. We use a normalization method of A.P.Norden [13], [19]. A. P. Shirokov in his work [20] constructed conformal models of Non-Euclidean spaces with this method.

In a projective space P_n a hypersurface X_{n-1} as an absolute is called *normalized* if with every point $Q \in X_{n-1}$ there is associated:
1) a line P_I which has the point Q as the only intersection with the tangent space T_{n-1},
2) a linear space P_{n-2} that belongs to T_{n-1}, but it does not contain the point Q. We call them *normals of first and second types*, P_I and P_{II}.

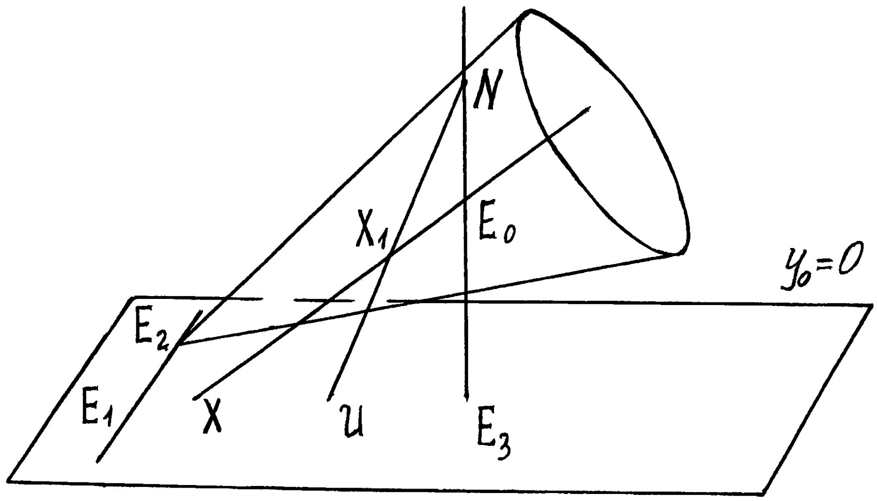
In order to have a polar normalization, P_I and P_{II} must be polar with respect to the absolute X_{n-1}.

We enlarge the semi-Euclidean space ${}_2E^3_1$ to a projective space P^3. Here ${}_kE^n_l$ denotes a n-dimensional semi-Euclidean space with the metric tensor of rank k, and l is the number of negative inertia index in a quadric form. We consider homogeneous coordinates $(y_0 : y_1 : y_2 : y_3)$ in P^3, where $x_i = \frac{y_i}{y_3}, i = 0, 1, 2$. Thus $S^2(1) : x_0^2 - x_1^2 = 1$ describes the hyperquadric in P^3:

$$y_0^2 - y_1^2 - y_3^2 = 0\,. \tag{21}$$

Here the projective basis (E_0, E_1, E_2, E_3) is chosen in the following way. The vertex E_0 of basis is inside the hyperquadric. The other vertices E_1, E_2, E_3 are on its polar plane, $y_0 = 0$. The line E_0E_3 crosses the hyperquadric at poles $N(1:0:0:1)$, $N'(1:0:0:-1)$. Vertices E_1, E_2 lie on the polar of the line E_0E_3. The vertex of the hyperquadric coincides with the vertex E_2.

The stereographic map of the projective plane $P^2 : y_0 = 0$ to the hyperquadric (21) from the pole $N(1:0:0:1)$ is shown on the picture. Let $U(0 : y_1 : y_2 : y_3) \in P^2$. If $y_3 = 0$, then the line UN belongs to the tangent plane $T_N : y_0 - y_3 = 0$ of the hyperquadric (21) at the point N and in this case the intersection point of the line UN with the hyperquadric is not uniquely determined. If $y_3 \neq 0$, then the intersection point of the line UN with the hyperquadric is unique. So, we choose the line $E_1E_2 : y_3 = 0$ as the line at infinity. In the area $y_3 \neq 0$ we consider the Cartesian coordinates $x_1 = \frac{y_1}{y_3}, x_2 = \frac{y_2}{y_3}$. Then the plane $\alpha : y_0 = 0, y_3 \neq 0$ becomes a plane with an affine structure A^2. It is possible to introduce the



structure of semi-Euclidean plane ${}_1E^2$ with the linear element

$$ds_0^2 = dx_1^2\,. \tag{22}$$

The hyperquadric and the plane α do not intersect or intersect in two imaginary parallel lines

$$x_1^2 = -1\,. \tag{23}$$

The restriction of the stereographic projection to the plane α maps the point $U(0 : x_1 : x_2 : 1)$ into the point X_1

$$X_1(-1 - x_1^2 : 2x_1 : 2x_2 : 1 - x_1^2)\,. \tag{24}$$

So, the Cartesian coordinates x_i can be used as the local coordinates at the hyperquadric except the point of its intersection with the tangent plane T_N.

We construct an autopolar normalization of the hyperquadric. As a normal of the first type we take lines with the fixed center E_0 and as a normal of the second type we take their polar lines which belong to the plane α and cross the vertex E_2 of the hyperquadric. The line E_0X_1 intersects the plane α at the point

$$X(0 : 2x_1 : 2x_2 : 1 - x_1^2)\,.$$

In this normalization the polar of the point X intersects the plane α on the normal P_{II}. Thus for any point X in the plane α there corresponds a line which does not cross this point. It means that the plane α is also normalized. The normalization of α is defined by an absolute quadric (23).

We consider the derivative equations of this normalization. If we take normals of the first type with fixed center E_0, then the derivative equations ([13], p.204)

have the form:

$$\begin{aligned} \partial_i X &= Y_i + l_i X \ , \\ \nabla_j Y_i &= l_j Y_i + p_{ji} X \ . \end{aligned} \tag{25}$$

The points X, Y_i, E_0 define a family of projective frames. Here Y_i are generating points of the normal P_{II}.

We can calculate the values (X, X), (X, Y_i) on the plane α using the quadric form, which is in the left part of equation (21). So, $(X, X) = -(1 + x_1^2)^2$.

Let us find coordinates of the metric tensor on the plane α. Hence, we take the Weierstrass standardization

$$(\widetilde{X}, \widetilde{X}) = -1, \quad \widetilde{X} = \frac{X}{1 + x_1^2} \, .$$

Then the coordinates of the metric tensor are the scalar products of partial derivatives $g_{ij} = -(\partial_i \widetilde{X}, \partial_j \widetilde{X})$:

$$(g_{ij}) = \begin{pmatrix} \frac{4}{(1+x_1^2)^2} & 0 \\ 0 & 0 \end{pmatrix} .$$

We got the conformal model of the polar normalized plane $\alpha : y_0 = 0, y_3 \neq 0$ with a linear element

$$ds^2 = \frac{dx_1^2}{(1 + x_1^2)^2} \, . \tag{26}$$

It means that this non-Euclidean plane is conformally equivalent to semi-Euclidean plane ${}_1E^2$.

The points X and Y_i are conjugated with respect to the polar (21) and $(X, Y_i) = 0$. From this equation and the derivative equations (25) we can get the non-zero connection coefficients:

$$\Gamma_{11}^1 = \Gamma_{12}^2 = \Gamma_{21}^2 = -\frac{2x_1}{1 + x_1^2} \, , \quad \Gamma_{11}^2 = \frac{2x_2}{1 + x_1^2} \, .$$

The sums $\Gamma_{ks}^s = \partial_k \ln \frac{c}{(1+x_1^2)^2}$ $(c = const)$ are gradients, so the connection is equiaffine. Curvature tensor has the following non-zero elements:

$$R_{121\cdot}{}^2 = -R_{211\cdot}{}^2 = -\frac{4}{(1 + x_1^2)^2} \, .$$

Ricci curvature tensor $R_{sk} = R_{isk\cdot}{}^i$ is symmetric: $R_{11} = \frac{4}{(1+x_1^2)^2}$. Metric g_{ij} and curvature $R_{rsk\cdot}{}^i$ tensors are covariantly constant in this connection: $\nabla_k g_{ij} = 0$, $\nabla_l R_{rsk\cdot}{}^i = 0$. The infinitesimal linear operators for the quadric are

$$\begin{cases} L_1 = y_0 \frac{\partial}{\partial y_1} + y_1 \frac{\partial}{\partial y_0} \\ L_2 = y_0 \frac{\partial}{\partial y_3} + y_3 \frac{\partial}{\partial y_0} \\ L_3 = y_1 \frac{\partial}{\partial y_3} - y_3 \frac{\partial}{\partial y_1} \end{cases} \tag{27}$$

Solving geodesic equations we find parametric solutions

$$\left\{\begin{array}{l} x_1 = \tan(\omega t + \phi) \\ x_2 = (c_1 e^{2i\omega t} + c_2 e^{-2i\omega t}) \sec^2(\omega t + \phi). \end{array}\right. \tag{28}$$

where c_1, c_2, ω, ϕ are integration constants. Eliminating the parameter t we can rewrite these equations in a simple form

$$x_2 = A(x_1^2 - 1) + Bx_1$$

where A and B are arbitrary constants. We see that the solution represents parabolas and lines in x_2x_1 plane.

Let us consider the bundle of this plane by the double numbers subalgebra. We write the equations of fibers of semi-Euclidean sphere $S^2(1)$ in homogeneous coordinates:

$$\left\{\begin{array}{l} (y_0 - y_1)v - y_2 = 0, \\ y_0^2 - y_1^2 - y_3^2 = 0. \end{array}\right. \tag{29}$$

This 1-parametric family of curves fibers the hyperquadric and it defines a bundle on it. The image of these fibers under the stereographic projection from the pole N to the plane α is:

$$x_2 = -v/2 \cdot (x_1 + 1)^2.$$

It is 1-parametric family of parabolas.

Remark

We would obtain the similar results for the space of right cosets by the Lie subgroup H_2 (it is the subgroup of invertible dual numbers) and the bundle of the group G by H_2. However, H_2 is a normal divisor of the group G. Therefore, the spaces of right and left cosets coincide.

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